ELSEVIER

Bull. Sci. math. 134 (2010) 549–560

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Lower bounds of Tian's invariant under toric invariances

Adnène Ben Abdesselem [a,*], Malek Filali [b]

[a] *Université Paris 6, U.F.R. 929, 4, place Jussieu, 75005, Paris, France*
[b] *L.I.M. Ecole Polytechnique de Tunisie, Tunisia*

Received 25 February 2010

Available online 13 April 2010

Abstract

On some toric Fano manifolds with metrics in the first Chern class, we show that a large family of smooth almost pluri-subharmonic functions (i.e. subharmonic with respect to the metric) with maximum equal to 0 admits a lower envelope. In our previous papers (A. Ben Abdesselem (2006) [4] and A. Ben Abdesselem and B. Dridi (2008) [5]) we established such envelopes when the functions considered are invariant under the action of a larger automorphisms group. Here we only consider the invariances due to the of the toric structure of the manifolds.

1. Introduction and background

This article presents further results on lower bounds of almost pluri-subharmonic functions on Fano manifolds obtained from $\mathbb{P}_m\mathbb{C}$, the projective complex space of complex dimension m (here we take $m > 1$). Fewer and fewer symmetries are taken into account in order to arrive eventually at a result concerning functions with the least possible invariance (see [3–5]).

Let $[z_0, \dots, z_m]$ denote the homogeneous coordinates in $\mathbb{P}_m\mathbb{C}$. We endow $\mathbb{P}_m\mathbb{C}$ with the Fubini–Study metric g whose components in the chart $\{[z_0, \dots, z_m] \in \mathbb{P}_m\mathbb{C}$ s.t. $z_0 \neq 0\}$ are given by

$$g_{\lambda\bar{\mu}} = \partial_{\lambda\bar{\mu}} \ln(1 + x_1 + \cdots + x_m)$$

where $x_i = |z_i|^2$ and $\partial_{\lambda\bar{\mu}} = \frac{\partial^2}{\partial z_\lambda \partial \bar{z}_\mu}$.

* Corresponding author.
E-mail address: benabdes@math.jussieu.fr (A. Ben Abdesselem).

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2010.04.001

Let $0 < p \leqslant m+1$ and n_j, $j \in \{1, \dots, p\}$ be fixed integers, we identify points of $\mathbb{P}_m\mathbb{C}$ as follows:

$$[Z_1, \dots, Z_p] = [z_0, \dots, z_{n_1-1}; z_{n_1}, \dots, z_{n_1+n_2-1}; \dots; \\ z_{n_1+\dots+n_{p-1}}, \dots, z_{n_1+\dots+n_p-1} = z_m],$$

where Z_k is the n_k-tuple: $Z_k = (z_{n_1+\dots+n_{k-1}}, \dots, z_{n_1+\dots+n_k-1}) \in \mathbb{C}^{n_k}$.

M is the blow-up of $\mathbb{P}_m\mathbb{C}$ over the subsets

$$\{[0^{[n_1]}; \dots; 0^{[n_{k-1}]}; z_{n_1+\dots+n_{k-1}}, \dots, z_{n_1+\dots+n_k-1}; 0^{[k+1]}; \dots; 0^{[p]}]\}$$

(where $0^{[n_k]} = (0, 0, \dots, 0) \in \mathbb{C}^{n_k}$) respectively identified to $\mathbb{P}_{n_1-1}\mathbb{C}, \dots, \mathbb{P}_{n_p-1}\mathbb{C}$.

M can be defined as the submanifold of $\mathbb{P}_m\mathbb{C} \times \mathbb{P}_{n_1-1}\mathbb{C} \times \dots \times \mathbb{P}_{n_p-1}\mathbb{C}$ described by the points $([Z_1, \dots, Z_p], [\zeta_0, \dots, \zeta_{n_1-1}], \dots, [\zeta_{n_1+\dots+n_{p-1}}, \dots, \zeta_{n_1+\dots+n_p-1} = \zeta_m]) \in \mathbb{P}_m\mathbb{C} \times \mathbb{P}_{n_1-1}\mathbb{C} \times \dots \times \mathbb{P}_{n_p-1}\mathbb{C}$ such that the vectors $(\zeta_{n_1+\dots+n_{k-1}}, \dots, \zeta_{n_1+\dots+n_k-1})$ and Z_k are proportional (let us note that these manifolds cannot carry Einstein–Kähler metric (see [7]) when the considered subsets have not the same dimension). We consider the projections $\pi_0, \pi_1, \dots, \pi_p$ of M respectively on $\mathbb{P}_m\mathbb{C}, \mathbb{P}_{n_1-1}\mathbb{C}, \dots, \mathbb{P}_{n_p-1}\mathbb{C}$.

Denoting by g_k the Fubini–Study metrics on $\mathbb{P}_{n_k-1}\mathbb{C}$ and by g_m that of $\mathbb{P}_m\mathbb{C}$, we define a metric $\tilde{g}$ on M by

$$\tilde{g} = p\pi_0^* g_m + (n_1 - 1)\pi_1^* g_1 + \dots + (n_p - 1)\pi_p^* g_p$$

whose components in the dense local chart of M

$$\{([1, z_1, \dots, z_m], [1, z_1, \dots, z_{n_1-1}], \dots, [z_{n_1+\dots+n_{p-1}}, \dots, z_m]), (z_1, \dots, z_m) \in \mathbb{C}^m \text{ and } \\ (z_{n_1+\dots+n_{k-1}}, \dots, z_{n_1+\dots+n_k-1}) \neq 0\}$$

are given by

$$\tilde{g}_{\lambda\overline{\mu}} = p\partial_{\lambda\overline{\mu}} \ln(1 + x_1 + \dots + x_m) + (n_1 - 1)\partial_{\lambda\overline{\mu}} \ln(1 + x_1 + \dots + x_{n_1-1}) + \dots \\ + (n_p - 1)\partial_{\lambda\overline{\mu}} \ln(x_{n_1+\dots+n_{p-1}} + \dots + x_m).$$

Let us recall that $\tilde{g}$ belongs to the first Chern class of M and consequently (M, g) is a Fano manifold.

Now let us consider the automorphisms group G on $\mathbb{P}_m\mathbb{C}$ spanned by automorphisms σ_{i_k,s_k}, $\tau_{l,\theta}$ defined $\forall i_k, s_k \in \{n_1 + \dots + n_{k-1}, \dots, n_1 + \dots + n_k - 1\}$, $l \in \{0, 1, \dots, m\}$ and $\theta \in [0, 2\pi]$ by

$$\sigma_{i_k,s_k}([z_0, \dots; z_{n_1+\dots+n_{k-1}}, \dots, z_{i_k}, \dots, z_{s_k}, \dots, z_{n_1+\dots+n_k-1}; \dots, z_m]) \\ = [z_0, \dots; z_{n_1+\dots+n_{k-1}}, \dots, z_{s_k}, \dots, z_{i_k}, \dots, z_{n_1+\dots+n_k-1}; \dots, z_m]$$

and

$$\tau_{l,\theta}([z_0, \dots, z_l, \dots, z_m]) = [z_0, \dots, z_l e^{i\theta}, \dots, z_m].$$

This group induces natural automorphisms groups on M which we denote again by G. We note that when $p = m+1$, then $\sigma_{i_k,s_k} = Id$ and therefore the functions are only invariant under the toric action.

Let us consider on $\mathbb{C}^{m+1} \setminus \bigcup_i \{z_i = 0\}$ the function $\psi = \inf(\psi_1, \dots, \psi_p)$, where

$$\psi_k = \ln \frac{(|z_{n_1+\dots+n_{k-1}}| \cdots |z_{n_1+\dots+n_k-1}|)^{2(m+1)/n_k}}{(|z_0|^2 + \dots + |z_m|^2)^{(m+1)}}.$$

These functions are zero-homogeneous in the components of $\mathbb{C}^{m+1}$. To describe the extremal function $\tilde{\psi}$ on M, we consider the functions $\tilde{\psi}_k$

$$\tilde{\psi}_k = \ln\left\{ \frac{(|z^{(0)}_{n_1+\cdots+n_{k-1}}|\cdots|z^{(0)}_{n_1+\cdots+n_k-1}|)^{\frac{2p}{n_k}}}{(|z^{(0)}_0|^2+\cdots+|z^{(0)}_m|^2)^p} \times \prod_{i=1}^{p} \frac{(|z^{(i)}_0|\cdots|z^{(i)}_{n_i-1}|)^{\frac{2(n_i-1)}{n_i}}}{(|z^{(i)}_0|^2+\cdots+|z^{(i)}_{n_i-1}|^2)^{n_i-1}} \right\}.$$

These functions are defined on

$$\left(\mathbb{C}^{m+1}\Big\backslash\bigcup_i\{z^{(0)}_i=0\}\right)\times\left(\mathbb{C}^{n_1}\Big\backslash\bigcup_j\{z^{(1)}_j=0\}\right)\times\cdots\times\left(\mathbb{C}^{n_p}\Big\backslash\bigcup_j\{z^{(p)}_j=0\}\right)$$

where $(z^{(0)}_i)_{0\leqslant i\leqslant m}, (z^{(1)}_i)_{0\leqslant i\leqslant n_1-1}, \ldots, (z^{(p)}_i)_{0\leqslant i\leqslant n_p-1}$ are respectively the coordinates on $\mathbb{C}^{m+1}$, $\mathbb{C}^{n_1}, \ldots, \mathbb{C}^{n_p}$, and are separately zero-homogeneous in the components of $\mathbb{C}^{m+1}$, $\mathbb{C}^{n_1}, \ldots, \mathbb{C}^{n_p}$. Consequently, they define functions on $\mathbb{P}_m\mathbb{C}\times\mathbb{P}_{n_1-1}\mathbb{C}\times\cdots\times\mathbb{P}_{n_p-1}\mathbb{C}$, and then, by restriction, on M. We set $\tilde{\psi}=\inf(\tilde{\psi}_1,\ldots,\tilde{\psi}_p)$.

2. Statement of results

Theorem 1. *Let $\varphi\in C^\infty(\mathbb{P}_m\mathbb{C})$ be a g-admissible and G-invariant function, verifying* $\sup\varphi=0$ *on $\mathbb{P}_m\mathbb{C}$. Then, we have $\varphi\geqslant\psi$.*

Let us recall that φ is said g-admissible when the matrix with entries $g_{\lambda\bar{\mu}}+\frac{\partial^2}{\partial z_\lambda\partial\bar{z}_\mu}$ is definite positive.

As an immediate corollary of this theorem we have a lower bound of Tian invariant introduced in [9] (see also [1,2,6]) given by:

Theorem 2. *For all $\alpha<\inf_{1\leqslant k\leqslant p}(\frac{n_k}{m+1})$, we have the Hörmander type inequality* (*see* [8])

$$\int\limits_{\mathbb{P}_m\mathbb{C}} \exp(-\alpha\varphi)\,dv\leqslant \text{Const.},$$

for all g-admissible and G-invariant function $\varphi\in C^\infty(\mathbb{P}_m\mathbb{C})$ satisfying $\sup\varphi=0$ *on $\mathbb{P}_m\mathbb{C}$. dv is the volume element on $\mathbb{P}_m\mathbb{C}$ with respect to the metric g.*

And finally

Theorem 3. *The inequality $\varphi\geqslant\tilde{\psi}$ holds, for all $\tilde{g}$-admissible and G-invariant function $\varphi\in C^\infty(M)$ satisfying* $\sup\varphi=0$ *on M.*

A similar corollary than the preceding one holds. We shall not give the proof of Theorem 3, which is similar to that of Theorem 1, but only outline its steps.

The next section is devoted to the proofs of the results announced in this introduction.

3. Proofs of the results

3.1. Proof of Theorem 1

The proof needs four lemmas. We shall use the invariance of the functions $\varphi([z_0,\ldots,z_m])$ by the group G defined in the introduction. We will consider the functions φ in Lemma 1 as functions of $([1, x_1, \ldots, x_m])$ where $x_i = |z_i|$, $i \in \{1, \ldots, m\}$ are real entries. In Lemma 2 we consider the functions φ as functions of $([x_0, \ldots, x_{n_1-1}, 1, x_{n_1+1}, \ldots, x_m])$ where $x_i = |z_i|$, $i \in \{0, \ldots, n_1-1, n_1+1, \ldots, m\}$.

Lemma 1. *Let $\varphi \in C^\infty(\mathbb{P}_m\mathbb{C})$ be a g-admissible, G-invariant function and let $x_i = |z_i| > 0$ for all i. Then*:

$$(\varphi - \psi)\big([1, x_1, \ldots, x_m]\big) \geqslant (\varphi - \psi)\big(\big[1, x_1, \ldots, x_{n_1-1}; \zeta_2^{[n_2]}, \ldots, \zeta_p^{[n_p]}\big]\big), \tag{1}$$

where $\zeta_k = (x_{n_1+\cdots+n_{k-1}} \times \cdots \times x_{n_1+\cdots+n_k-1})^{1/n_k}$, $2 \leqslant k \leqslant p$ *and* $\zeta^{[l]} = (\zeta, \ldots, \zeta) \in \mathbb{C}^l$.

Proof. Let us show that

$$(\varphi - \psi)\big([1, x_1, \ldots, x_m]\big) \geqslant (\varphi - \psi)\big(\big[1, x_1, \ldots, x_{n_1-1}; \zeta_2^{[n_2]}; x_{n_1+n_2+n_3}, \ldots, x_m\big]\big),$$

the proof is identical for each of the other groups of variables. The result is obtained by induction. Suppose that for $n_1 \leqslant j < n_1 + n_2 - 1$ and for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$ with $x_i > 0$ we have

$$\begin{aligned}
&(\varphi - \psi)\big([1, x_1, \ldots, x_m]\big) \\
&\quad \geqslant (\varphi - \psi)\big(\big[1, x_1, \ldots, x_{n_1-1}; (x_{n_1} \cdots x_j)^{\frac{1}{j-n_1+1}}, \ldots, (x_{n_1} \cdots x_j)^{\frac{1}{j-n_1+1}}, \\
&\qquad x_{j+1}, \ldots, x_{n_1+n_2-1}; x_{n_1+n_2}, \ldots, x_m\big]\big),
\end{aligned} \tag{2}$$

which is obviously verified for $j = n_1$. Now, suppose that inequality (2) is not satisfied at the step $j + 1$. Then, there would be a point $(x_1^0, \ldots, x_m^0) \in \mathbb{R}^m$ with $x_i^0 > 0$ for all i, such that

$$\begin{aligned}
&(\varphi - \psi)\big(\big[1, x_1^0, \ldots, x_m^0\big]\big) \\
&\quad < (\varphi - \psi)\big(\big[1, x_1^0, \ldots, x_{n_1-1}^0; \big(x_{n_1}^0 \cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+2}}, \ldots, \big(x_{n_1}^0 \cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+2}}, x_{j+2}^0, \ldots, \\
&\qquad x_{n_1+n_2-1}^0; x_{n_1+n_2}^0, \ldots, x_m^0\big]\big).
\end{aligned} \tag{3}$$

Using the continuity of $(\varphi - \psi)$, we can suppose (by a slight change of coordinates if necessary), that the ζ_i defined in $([1, x_1^0, \ldots, x_m^0])$ of inequality (3) are pairwise different; this is a property which we shall use later. Now, using the G-invariance of φ, we can suppose that $x_{n_1}^0 \leqslant \cdots \leqslant x_{n_1+n_2-1}^0$. On the other hand, the G-invariance of φ and the hypothesis (2) at the points

$$\big[1, x_1^0, \ldots, x_{n_1-1}^0; x_{n_1}^0, x_{n_1+1}^0, \ldots, x_j^0, x_{j+1}^0, x_{j+2}^0, \ldots, x_m^0\big]$$

and

$$\big[1, x_1^0, \ldots, x_{n_1-1}^0; x_{n_1+1}^0, \ldots, x_j^0, x_{j+1}^0, x_{n_1}^0, x_{j+2}^0, \ldots, x_m^0\big]$$

lead to

$$\begin{aligned}
&(\varphi - \psi)\big(\big[1, x_1^0, \ldots, x_m^0\big]\big) \\
&\quad \geqslant (\varphi - \psi)\big(\big[1, x_1^0, \ldots, x_{n_1-1}^0; \big(x_{n_1}^0 \cdots x_j^0\big)^{\frac{1}{j-n_1+1}}, \ldots, \big(x_{n_1}^0 \cdots x_j^0\big)^{\frac{1}{j-n_1+1}}, x_{j+1}^0, \ldots, \\
&\qquad x_{n_1+n_2-1}^0; x_{n_1+n_2}^0, \ldots, x_m^0\big]\big)
\end{aligned} \tag{4}$$

and

$$\begin{aligned}&(\varphi-\psi)\big(\big[1,x_1^0,\dots,x_m^0\big]\big)\\&\quad\geqslant(\varphi-\psi)\big(\big[1,x_1^0,\dots,x_{n_1-1}^0;\big(x_{n_1+1}^0\cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+1}},\dots,\big(x_{n_1+1}^0\cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+1}},\\&\qquad x_{n_1}^0,x_{j+2}^0,\dots,x_{n_1+n_2-1}^0;x_{n_1+n_2}^0,\dots,x_m^0\big]\big).\end{aligned}\tag{5}$$

Let us consider the curve C given by the equation

$$t^{j-n_1+1}x=x_{n_1}^0\cdots x_{j+1}^0$$

in the real plane $\{[1,x_1^0,\dots,x_{n_1-1}^0,t,\dots,t,x,x_{j+2}^0,\dots,x_m^0]\}$, parametrized by the variables t and x. The points

$$P_1=\big[1,x_1^0,\dots,x_{n_1-1}^0;\big(x_{n_1}^0\cdots x_j^0\big)^{\frac{1}{j-n_1+1}},\dots,\big(x_{n_1}^0\cdots x_j^0\big)^{\frac{1}{j-n_1+1}},x_{j+1}^0,x_{j+2}^0,\dots,x_m^0\big]$$

and

$$\begin{aligned}P_2=\big[&1,x_1^0,\dots,x_{n_1-1}^0;\big(x_{n_1+1}^0\cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+1}},\dots,\big(x_{n_1+1}^0\cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+1}},\\&x_{n_1}^0,x_{j+2}^0,\dots,x_m^0\big]\end{aligned}$$

belong to the curve C. The real numbers x_i^0 for $n_1\leqslant i\leqslant j+1$ are not all equal, because in this case (3) would be an equality.

Taking into account that we have chosen $x_{n_1}^0\leqslant\cdots\leqslant x_{j+1}^0$, the points P_1 and P_2 are on different sides of the diagonal $t=x$ of the plane described above. Note that the curve C intersects this diagonal at the point

$$P_3=\big[1,x_1^0,\dots,x_{n_1-1}^0;\big(x_{n_1}^0\cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+2}},\dots,\big(x_{n_1}^0\cdots x_{j+1}^0\big)^{\frac{1}{j-n_1+2}},x_{j+2}^0,\dots,x_m^0\big]$$

which appears in inequality (3). On the other hand, using (3), (4) and (5), we obtain

$$(\varphi-\psi)(P_3)>(\varphi-\psi)(P_1)\quad\text{and}\quad(\varphi-\psi)(P_3)>(\varphi-\psi)(P_2),$$

which proves that the function $(\varphi-\psi)$ reaches a local maximum on the curve C. Consequently, the restriction of the G-invariant function $(\varphi-\psi)$ to the holomorphic curve (still denoted by C) $\xi^{j-n_1+1}z=x_{n_1}^0\cdots x_{j+1}^0$ of the complex dimensional 2-plane

$$\big[1,x_1^0,\dots,x_{n_1-1}^0;\xi,\dots,\xi,z,x_{j+2}^0,\dots,x_m^0\big]$$

reaches a local maximum at a point $P=C(\zeta)$. We set $C(\zeta)=[1,C^1(\zeta),\dots,C^m(\zeta)]$, $\dot C^\lambda(\xi)=\frac{dC^\lambda}{d\xi}(\xi)$ and $\dot C^{\overline{\mu}}(\xi)=\overline{\dot C^{\mu}(\xi)}$.

Taking into account that we chose the point $[1,x_1^0,\dots,x_m^0]$ such that the ζ_i are all distinct at this point, the equation of the curve C and the definitions of ψ_i show that for all points of C and for all $i\neq l$,

$$\begin{aligned}&\psi_i\big(\big[1,x_1^0,\dots,x_{n_1-1}^0;\xi,\dots,\xi,z,x_{j+2}^0,\dots,x_m^0\big]\big)\\&\quad\neq\psi_l\big(\big[1,x_1^0,\dots,x_{n_1-1}^0;\xi,\dots,\xi,z,x_{j+2}^0,\dots,x_m^0\big]\big).\end{aligned}$$

Thus, we can suppose that $\psi=\psi_1$ in a neighborhood of P, the proof being exactly the same if we suppose $\psi=\psi_i$ for $i\in\{2,\dots,p\}$ in this neighborhood. Then,

$$\frac{\partial^2}{\partial\xi\partial\overline{\xi}}\big\{(\varphi-\psi_1)\big(C(\zeta)\big)\big\}=\frac{\partial^2(\varphi-\psi_1)}{\partial z_\lambda\partial\overline{z}_\mu}\big(C(\zeta)\big)\dot C^\lambda(\zeta)\dot C^{\overline{\mu}}(\zeta)$$

is negative. Since $-\frac{\partial^2\psi_1}{\partial z_\lambda \partial \bar{z}_\mu} = g_{\lambda\bar{\mu}}$, the matrix of the Hermitian form

$$\left(g_{\lambda\bar{\mu}} + \frac{\partial^2\varphi}{\partial z_\lambda \partial \bar{z}_\mu}\right)_{\lambda,\mu} = \left(\frac{\partial^2(\varphi-\psi_1)}{\partial z_\lambda \partial \bar{z}_\mu}\right)_{\lambda,\mu}$$

is negative at $P = C(\zeta)$. This yields a contradiction with the g-admissibility of φ at P. Thus, inequality (2) holds at step $j+1$, and Lemma 1 is proven. □

Lemma 2. *Let $\varphi \in C^\infty(\mathbb{P}_m\mathbb{C})$ be a g-admissible, G-invariant function. Then in the chart $\{z_{n_1} \neq 0\}$ we have that for all $x_i = |z_i| > 0$,*

$$\begin{aligned}&(\varphi-\psi)\big([x_0, x_1, \ldots, x_{n_1-1}; 1, x_{n_1+1}, \ldots, x_m]\big)\\ &\quad \geqslant (\varphi-\psi)\big([\eta, \eta, \ldots, \eta; 1, x_{n_1+1}, \ldots, x_m]\big),\end{aligned} \tag{6}$$

where $\eta = (x_0 x_1 \cdots x_{n_1-1})^{1/n_1}$.

Proof. As in Lemma 1 we proceed by induction. Suppose that, for $0 \leqslant j < n_1$ and for all $(x_0, \ldots, x_{n_1-1}; x_{n_1+1}, \ldots, x_m) \in \mathbb{R}^m$ with $x_i > 0$, we have

$$\begin{aligned}&(\varphi-\psi)\big([x_0, \ldots, x_{n_1-1}, 1, x_{n_1+1}, \ldots, x_m]\big)\\ &\quad \geqslant (\varphi-\psi)\big(\big[(x_0\cdots x_j)^{\frac{1}{j+1}}, \ldots, (x_0\cdots x_j)^{\frac{1}{j+1}}, x_{j+1}, \ldots, x_{n_1-1};\\ &\qquad 1, x_{n_1+1}, \ldots, x_m\big]\big),\end{aligned} \tag{7}$$

which is obviously verified for $j=0$. If inequality (7) were not satisfied at step $j+1$, then there would be a point $(x_0^0, \ldots, x_{n_1-1}^0; x_{n_1+1}^0, \ldots, x_m^0) \in \mathbb{R}^m$ with $x_i^0 > 0$ for all i, such that

$$\begin{aligned}&(\varphi-\psi)\big(\big[x_0^0, \ldots, x_{n_1-1}^0; 1, x_{n_1+1}^0, \ldots, x_m^0\big]\big)\\ &\quad < (\varphi-\psi)\big(\big[\big(x_0^0\cdots x_{j+1}^0\big)^{\frac{1}{j+2}}, \ldots, \big(x_0^0\cdots x_{j+1}^0\big)^{\frac{1}{j+2}}, x_{j+2}^0, \ldots, x_{n_1-1}^0;\\ &\qquad 1, x_{n_1+1}^0, \ldots, x_m^0\big]\big).\end{aligned} \tag{8}$$

As in Lemma 1, we can suppose that the point $[x_0^0, \ldots, x_{n_1-1}^0; 1, x_{n_1+1}^0, \ldots, x_m^0]$ satisfies $(x_0^0\cdots x_{n_1-1}^0)^{1/n_1} \neq \zeta_i$ for $i \geqslant 2$ and $x_0^0 \leqslant \cdots \leqslant x_{n_1}^0$. On the other hand, taking into account the G-invariance of φ and the hypothesis (7) at points

$$\big[x_0^0, x_1^0, \ldots, x_j^0, x_{j+1}^0, \ldots, x_{n_1-1}^0; 1, x_{n_1+1}^0, \ldots, x_m^0\big]$$

and

$$\big[x_1^0, \ldots, x_{j+1}^0, x_0^0, x_{j+2}^0, \ldots, x_{n_1-1}^0; 1, x_{n_1+1}^0, \ldots, x_m^0\big],$$

we obtain that

$$\begin{aligned}&(\varphi-\psi)\big(\big[x_0^0, \ldots, x_{n_1-1}^0; 1, x_{n_1+1}^0, \ldots, x_m^0\big]\big)\\ &\quad \geqslant (\varphi-\psi)\big(\big[\big(x_0^0\cdots x_j^0\big)^{\frac{1}{j+1}}, \ldots, \big(x_0^0\cdots x_j^0\big)^{\frac{1}{j+1}}, x_{j+1}^0, \ldots, x_{n_1-1}^0;\\ &\qquad 1, x_{n_1+1}^0, \ldots, x_m^0\big]\big)\end{aligned} \tag{9}$$

and

$$\begin{aligned}&(\varphi-\psi)\big(\big[x_1^0,\dots,x_{j+1}^0,x_0^0,\dots,x_{n_1-1}^0;1,x_{n_1+1}^0,\dots,x_m^0\big]\big)\\&\quad\geqslant(\varphi-\psi)\big(\big[\big(x_1^0\cdots x_{j+1}^0\big)^{\frac{1}{j+1}},\dots,\big(x_1^0\cdots x_{j+1}^0\big)^{\frac{1}{j+1}},x_0^0,x_{j+2}^0,\dots,x_{n_1-1}^0;\\&\qquad 1,x_{n_1+1}^0,\dots,x_m^0\big]\big).\end{aligned}\tag{10}$$

Let us consider the curve C given by the equation

$$t^{j+1}x=x_0^0\cdots x_{j+1}^0$$

in the real plane $\{[t,\dots,t,x,x_{j+2}^0,\dots,x_{n_1-1}^0;1,x_{n_1+1}^0,\dots,x_m^0]\}$, parametrized by the variables t and x. The points

$$Q_1=\big[\big(x_0^0\cdots x_j^0\big)^{\frac{1}{j+1}},\dots,\big(x_0^0\cdots x_j^0\big)^{\frac{1}{j+1}},x_{j+1}^0,x_{j+2}^0,\dots,x_{n_1-1}^0;1,x_{n_1+1}^0,\dots,x_m^0\big]$$

and

$$Q_2=\big[\big(x_1^0\cdots x_{j+1}^0\big)^{\frac{1}{j+1}},\dots,\big(x_1^0\cdots x_{j+1}^0\big)^{\frac{1}{j+1}},x_0^0,x_{j+2}^0,\dots,x_{n_1-1}^0;1,x_{n_1+1}^0,\dots,x_m^0\big]$$

belong to C. On the other hand, the real numbers x_i^0 for $0\leqslant i\leqslant p+1$ are not all equal, because otherwise (8) would be an equality. Then, since we chose $x_0^0\leqslant\cdots\leqslant x_{j+1}^0$, the distinct points Q_1 and Q_2 are on different sides of the diagonal $t=x$ of the previous plane. The curve C intersects the diagonal at

$$Q_3=\big[\big(x_0^0\cdots x_{j+1}^0\big)^{\frac{1}{j+2}},\dots,\big(x_0^0\cdots x_{j+1}^0\big)^{\frac{1}{j+2}},x_{j+2}^0,\dots,x_{n_1-1}^0;1,x_{n_1+1}^0,\dots,x_m^0\big]$$

which appears in inequality (8). The relations (8), (9) and (10) give

$$(\varphi-\psi)(Q_3)>(\varphi-\psi)(Q_1)\quad\text{and}\quad(\varphi-\psi)(Q_3)>(\varphi-\psi)(Q_2),$$

which proves that $(\varphi-\psi)$ reaches a local maximum on the curve C.

Since we choose the point $[x_0^0,\dots,x_{n_1-1}^0;1,x_{n_1+1}^0,\dots,x_m^0]$ such that $(x_0^0\cdots x_{n_1-1}^0)^{1/n_1}\neq\zeta_i$, for $i\geqslant 2$, the conclusion follows as in Lemma 1, by contradicting the g-admissibility of φ in Q. □

As a corollary of Lemmas 1 and 2 we have

Lemma 3. *Let $\varphi\in C^\infty(\mathbb{P}_m\mathbb{C})$ be a g-admissible and G-invariant function. Then for all $x_i=|z_i|>0$, we have that*

$$\begin{aligned}&(\varphi-\psi)\big([1,x_1,\dots,x_{n_1-1};x_{n_1},x_{n_1+1},\dots,x_m]\big)\\&\quad\geqslant(\varphi-\psi)\big(\big[1^{[n_1]};\eta_2^{[n_2]},\dots,\eta_p^{[n_p]}\big]\big),\end{aligned}\tag{11}$$

where $\eta_i=\frac{\zeta_i}{\eta}$, ζ_i and η defined in the two previous lemmas.

Proof. According to Lemma 1, we have

$$\begin{aligned}&(\varphi-\psi)\big([1,x_1,\dots,x_{n_1-1};x_{n_1},x_{n_1+1},\dots,x_m]\big)\\&\quad\geqslant(\varphi-\psi)\big(\big[1,x_1,\dots,x_{n_1-1};\zeta_2^{[n_2]},\dots,\zeta_p^{[n_p]}\big]\big)\\&\quad=(\varphi-\psi)\bigg(\bigg[\frac{1}{\zeta_2},\frac{x_1}{\zeta_2},\dots,\frac{x_{n_1-1}}{\zeta_2};1^{[n_2]};\frac{\zeta_3}{\zeta_2}^{[n_3]};\dots;\frac{\zeta_p}{\zeta_2}^{[n_p]}\bigg]\bigg).\end{aligned}$$

Since the (n_1+1)-th homogeneous component of this last point is 1, Lemma 2 gives

$$\begin{aligned}
&(\varphi-\psi)\big([1,x_1,\dots,x_{n_1-1};x_{n_1},x_{n_1+1},\dots,x_m]\big)\\
&\quad\geqslant(\varphi-\psi)\left(\left[\frac{\eta}{\zeta_2}^{[n_1]},1^{[n_2]};\frac{\zeta_3}{\zeta_2}^{[n_3]};\dots;\frac{\zeta_p}{\zeta_2}^{[n_p]}\right]\right)\\
&\quad=(\varphi-\psi)\big(\big[\eta^{[n_1]},\zeta_2^{[n_2]};\zeta_3^{[n_3]};\dots;\zeta_p^{[n_p]}\big]\big)\\
&\quad=(\varphi-\psi)\left(\left[1^{[n_1]},\frac{\zeta_2}{\eta}^{[n_2]};\frac{\zeta_3}{\eta}^{[n_3]};\dots;\frac{\zeta_p}{\eta}^{[n_p]}\right]\right),
\end{aligned}$$

and inequality (11) holds. □

Lemma 4. *Given a g-admissible, G-invariant function $\varphi\in C^\infty(\mathbb{P}_m\mathbb{C})$ verifying* $\sup=0$ *on* $\mathbb{P}_m\mathbb{C}$, *then for all $\eta_i>0$, we have*

$$(\varphi-\psi)\big(\big[1^{[n_1]},\eta_2^{[n_2]};\eta_3^{[n_3]};\dots;\eta_p^{[n_p]}\big]\big)\geqslant 0. \tag{12}$$

Proof. We consider the set of the points $P_0\in\mathbb{P}_m\mathbb{C}$ where φ reaches its maximum (equal to zero). Using the G-invariance of φ, we can write P_0 as

$$P_0=\big[y_0^0,\dots,y_m^0\big],$$

where y_k^0 are positive real numbers verifying $y_0^0\geqslant y_1^0\geqslant\cdots\geqslant y_{n_1-1}^0$ and $\forall k\in\{2,\dots,p-1\}$, $y^0_{n_1+\cdots+n_k}\geqslant\cdots\geqslant y^0_{n_1+\cdots+n_{k+1}-1}$.

Then, in the chart $\{z_{n_1+\cdots+n_i}\neq 0\}$, we can write P_0 as

$$P_0=\big[x_0^0,\dots,x_{n_1-1}^0;\dots;x^0_{n_1+\cdots+n_{i-2}},\dots,x^0_{n_1+\cdots+n_{i-1}-1};1,x^0_{n_1+\cdots+n_{i-1}+1},\dots,x_m^0\big]$$

where x_k^0 are positive real numbers verifying $x_0^0\geqslant\cdots\geqslant x^0_{n_1-1}$, $x^0_{n_1}\geqslant\cdots\geqslant x^0_{n_1+n_2-1},\dots,1>x^0_{n_1+\cdots+n_{i-1}+1}\geqslant\cdots\geqslant x^0_{n_1+\cdots+n_i-1},\dots$, and, finally, $x^0_{n_1+\cdots+n_{p-1}}\geqslant\cdots\geqslant x^0_{n_1+\cdots+n_p-1}=x^0_m$.

Arguing by contradiction, we suppose that there exists a point

$$P_1=\big[\eta_1^{[n_1]};\dots;\eta_{i-1}^{[n_{i-1}]};1^{[n_i]};\eta_{i+1}^{[n_{i+1}]},\dots,\eta_p^{[n_p]}\big]$$

such that $\eta_k>0$ pour $k\in\{1,\dots,i-1,i+1,\dots,p\}$, and

$$(\varphi-\psi)(P_1)<0. \tag{13}$$

Then, we distinguish two cases:

• <u>1-st case</u>:

Suppose that $x_0^0<\eta_1$, $x^0_{n_1}<\eta_2$, $\dots$, $x^0_{n_1+\cdots+n_{i-2}}<\eta_{i-1}$, $x^0_{n_1+\cdots+n_i}<\eta_{i+1}$, $\dots$, $x^0_{n_1+\cdots+n_{p-1}}<\eta_p$.

Then, we introduce the auxiliary function

$$\psi^0_{i-1}=\ln\frac{|z_{n_1+\cdots+n_{i-1}}|^{2(m+1)}}{(|z_0|^2+\cdots+|z_m|^2)^{m+1}}.$$

Since $\varphi\leqslant 0$, we obtain that

$$\big(\varphi-\psi^0_{i-1}\big)\big([0,\dots,0,1,0,\dots,0]\big)=\varphi\big([0,\dots,0,1,0,\dots,0]\big)\leqslant 0 \tag{14}$$

where the 1 is placed at the $(n_1+\cdots+n_{i-1})$-th coordinate.

Moreover, taking into account that $\varphi(P_0)=0$ and $\psi^0_{i-1}\leqslant 0$, we also obtain that

$$\big(\varphi-\psi^0_{i-1}\big)(P_0)\geqslant 0. \tag{15}$$

If $P_0\neq[0,\dots,0,1,0,\dots,0]$, then $\psi^0_{i-1}(P_0)<0$, and inequality (15) is strict. If $P_0=[0,\dots,0,1,0,\dots,0]$, we can choose another point P in a neighborhood of P_0, such that $(\varphi-\psi^0_{i-1})(P)>0$. Indeed, if $(\varphi-\psi^0_{i-1})\leqslant 0$ in any neighborhood of P_0, since $(\varphi-\psi^0_{i-1})(P_0)=0$, then $(\varphi-\psi^0_{i-1})$ reaches a local maximum at P_0, which contradicts the admissibility of φ at this point (recall that $\partial_{\lambda\overline{\mu}}(\varphi-\psi^0_{i-1})(P_0)=(g_{\lambda\overline{\mu}}+\partial_{\lambda\overline{\mu}}\varphi)(P_0))$. In conclusion, we deduce that there exists a point

$$P'_0=[a_0,\dots,a_{n_1-1};\dots;a_{n_1+\dots+n_{i-2}},\dots,a_{n_1+\dots+n_{i-1}-1};1,a_{n_1+\dots+n_{i-1}+1},\dots,a_m]$$

satisfying

$$(\varphi-\psi_0)\big(P'_0\big)>0. \tag{16}$$

By the continuity and G-invariance of φ, we can suppose that $\eta_1>a_1\geqslant\dots\geqslant a_{n_1-1}$ and $\eta_{i-1}>a_{n_1+\dots+n_{i-2}}\geqslant\dots\geqslant a_{n_1+\dots+n_{i-1}-1}$, $1>a_{n_1+\dots+n_{i-1}+1}\geqslant\dots\geqslant a_{n_1+\dots+n_i-1}$, $\eta_{i+1}>a_{n_1+\dots+n_i}\geqslant\dots\geqslant a_{n_1+\dots+n_{i+1}-1}$, $\dots$, $\eta_p>a_{n_1+\dots+n_{p-1}}\geqslant\dots\geqslant a_{n_1+\dots+n_p-1}=a_m$. On the other hand, inequality (13) and the definitions of P_1, ψ^0_{i-1}, ψ_i and $\psi=\inf(\psi_1,\dots,\psi_p)$ imply that

$$\big(\varphi-\psi^0_{i-1}\big)(P_1)=(\varphi-\psi_i)(P_1)\leqslant(\varphi-\psi)(P_1)<0. \tag{17}$$

The curve

$$\begin{aligned}[0,1]\ni t\to\Big[&\eta_1 t^{\frac{\ln(a_0/\eta_1)}{\ln\alpha}},\dots,\eta_1 t^{\frac{\ln(a_{n_1-1}/\eta_1)}{\ln\alpha}};\dots;\eta_{i-1}t^{\frac{\ln(a_{n_1+\dots+n_{i-2}}/\eta_{i-1})}{\ln\alpha}},\\ &\dots,\eta_{i-1}t^{\frac{\ln(a_{n_1+\dots+n_{i-1}-1}/\eta_{i-1})}{\ln\alpha}},1,t,t^{\ln(a_{n_1+\dots+n_{i-1}+2})/\ln\alpha},\dots,t^{\ln(a_{n_1+\dots+n_i-1})/\ln\alpha};\\ &\eta_{i+1}t^{\frac{\ln(a_{n_1+\dots+n_i}/\eta_{i+1})}{\ln\alpha}},\dots,\eta_{i+1}t^{\frac{\ln(a_{n_1+\dots+n_{i+1}-1}/\eta_{i+1})}{\ln\alpha}};\dots;\eta_p t^{\frac{\ln(a_{n_1+\dots+n_{p-1}}/\eta_p)}{\ln\alpha}},\dots,\\ &\eta_p t^{\frac{\ln(a_{n_1+\dots+n_p-1}/\eta_p)}{\ln\alpha}}\Big]\end{aligned}$$

(where $\alpha=a_{n_1+\dots+n_{i-1}+1}$) passes through the points $[0,\dots,0,1,0,\dots,0]$ at $t=0$, P'_0 at $t=\alpha$, and, finally, P_1 at $t=1$. At these points, using (14), (16) and (17), we deduce that $(\varphi-\psi^0_{i-1})$ is respectively negative, positive and negative. The invariance by multiplication with $\exp(i\theta)$ allows us to deduce that $(\varphi-\psi^0_{i-1})$ reaches a maximum on the holomorphic curve given by the complexification of the curve described above. This is in contradiction with the admissibility of φ.

• 2-nd case.

Now suppose that there exists $j\neq i$ such that

$$x^0_{n_1+\dots+n_j-1}\geqslant\eta_j.$$

We consider a neighborhood of P_0, where we can suppose that all $x^0_k\neq 0$. We set then

$$\alpha_0=\frac{\ln(x^0_0/\eta_1)}{\ln(x^0_{n_1+\dots+n_{i-1}+1})},\dots,\alpha_{n_1-1}=\frac{\ln(x^0_{n_1-1}/\eta_1)}{\ln(x^0_{n_1+\dots+n_{i-1}+1})};\dots;$$

$$\alpha_{n_1+\dots+n_{i-2}}=\frac{\ln(x^0_{n_1+\dots+n_{i-2}}/\eta_{i-1})}{\ln(x^0_{n_1+\dots+n_{i-1}+1})},\dots,\alpha_{n_1+\dots+n_{i-1}-1}=\frac{\ln(x^0_{n_1+\dots+n_{i-1}-1}/\eta_{i-1})}{\ln(x^0_{n_1+\dots+n_{i-1}+1})};$$

for the $(n_1+\cdots+n_{i-1})$-th index we set

$$\alpha_{n_1+\cdots+n_{i-1}}=\frac{\ln(x^0_{n_1+\cdots+n_{i-1}})}{\ln(x^0_{n_1+\cdots+n_{i-1}+1})},\alpha_{n_1+\cdots+n_{i-1}+1}=\frac{\ln(x^0_{n_1+\cdots+n_{i-1}+1})}{\ln(x^0_{n_1+\cdots+n_{i-1}+1})}=1,\dots,$$
$$\alpha_{n_1+\cdots+n_i-1}=\frac{\ln(x^0_{n_1+\cdots+n_i-1})}{\ln(x^0_{n_1+\cdots+n_{i-1}+1})},$$

and then we begin again with

$$\alpha_{n_1+\cdots+n_i}=\frac{\ln(x^0_{n_1+\cdots+n_i}/\eta_{i+1})}{\ln(x^0_{n_1+\cdots+n_{i-1}+1})},\dots,\alpha_{n_1+\cdots+n_{i+1}-1}=\frac{\ln(x^0_{n_1+\cdots+n_{i+1}-1}/\eta_{i+1})}{\ln(x^0_{n_1+\cdots+n_{i-1}+1})};\dots;$$
$$\alpha_{n_1+\cdots+n_{p-1}}=\frac{\ln(x^0_{n_1+\cdots+n_{p-1}}/\eta_p)}{\ln(x^0_{n_1+\cdots+n_{i-1}+1})},\dots,\alpha_{n_1+\cdots+n_{p-1}}=\frac{\ln(x^0_{n_1+\cdots+n_p-1}/\eta_p)}{\ln(x^0_{n_1+\cdots+n_{i-1}+1})}.$$

Because of the assumption that it exists $j\neq i$ such that $x^0_{n_1+\cdots+n_j-1}\geqslant\eta_j$, some of the powers α_k are negative. Let us call β the smallest of these, which corresponds to some $\alpha_{n_1+\cdots+n_{j_0}-1}$. For this index j_0, let us consider the auxiliary function

$$\psi^0_{j_0}=\ln\frac{|z_{n_1+\cdots+n_{j_0}-1}|^{2(m+1)}}{(|z_0|^2+\cdots+|z_m|^2)^{m+1}}.$$

We have

$$\big(\varphi-\psi^0_{j_0}\big)(P_0)>0. \tag{18}$$

Setting

$$\begin{aligned}P_\varepsilon=\big[&\eta_1\varepsilon^{\alpha_0},\dots,\eta_1\varepsilon^{\alpha_{n_1-1}};\dots;\eta_{i-1}\varepsilon^{\alpha_{n_1+\cdots+n_{i-2}}},\dots,\eta_{i-1}\varepsilon^{\alpha_{n_1+\cdots+n_{i-1}-1}};\\&1,\varepsilon,\varepsilon^{\alpha_{n_1+\cdots+n_{i-1}+2}},\dots,\varepsilon^{\alpha_{n_1+\cdots+n_i-1}}\eta_{i+1}\varepsilon^{\alpha_{n_1+\cdots+n_i}},\dots,\eta_{i+1}\varepsilon^{\alpha_{n_1+\cdots+n_{i+1}-1}};\\&\dots;\eta_p\varepsilon^{\alpha_{n_1+\cdots+n_{p-1}}},\dots,\eta_p\varepsilon^{\alpha_{n_1+\cdots+n_p-1}}\big],\end{aligned}$$

we have that

$$\psi^0_{j_0}(P_\varepsilon)=\ln\frac{(\eta_{j_0}\varepsilon^\beta)^{2(m+1)}}{N_\varepsilon^{(m+1)}},$$

where

$$\begin{aligned}N_\varepsilon=\eta_1^2&\big(\varepsilon^{2\alpha_0}+\cdots+\varepsilon^{2\alpha_{n_1-1}}\big)+\cdots+\eta_{i-1}^2\big(\varepsilon^{2\alpha_{n_1+\cdots+n_{i-2}}}+\cdots+\varepsilon^{2\alpha_{n_1+\cdots+n_i-1}}\big)\\&+1+\varepsilon^2+\cdots+\varepsilon^{2\alpha_{n_1+\cdots+n_i-1}}+\eta_{i+1}^2\big(\varepsilon^{2\alpha_{n_1+\cdots+n_i}}+\cdots+\varepsilon^{2\alpha_{n_1+\cdots+n_{i+1}-1}}\big)\\&+\cdots+\eta_p^2\big(\varepsilon^{2\alpha_{n_1+\cdots+n_{p-1}}}+\cdots+\varepsilon^{2\alpha_m}\big).\end{aligned}$$

When ε tends to 0, we obtain that

$$\begin{aligned}\lim_{\varepsilon\to0}\psi^0_{j_0}(P_\varepsilon)&=\lim_{\varepsilon\to0}\ln\frac{(\eta_{j_0}\varepsilon^\beta)^{2(m+1)}}{N_\varepsilon^{(m+1)}}\\&=\lim_{t\to\infty}\ln\frac{(\eta_{j_0}t^{-\beta})^{2(m+1)}}{[(\eta_{j_0}t^{-\beta})^2]^{(m+1)}}=\ln1=0,\end{aligned}$$

$-\beta$ being the largest power appearing at the denominator. Taking in account that $\varphi(P_\varepsilon) \leqslant 0$, as well as (18), there exists ε_0 such that

$$\big(\varphi - \psi^0_{j_0}\big)(P_{\varepsilon_0}) \leqslant -\psi^0_{j_0}(P_{\varepsilon_0}) < \big(\varphi - \psi^0_{j_0}\big)(P_0). \tag{19}$$

On the other hand, the inequality (13), the definitions of P_1, $\psi^0_{j_0}$, ψ_{j_0} and the fact that $\psi = \inf(\psi_1, \dots, \psi_p)$ imply that

$$\big(\varphi - \psi^0_{j_0}\big)(P_1) = (\varphi - \psi_{j_0})(P_1) \leqslant (\varphi - \psi)(P_1) < 0. \tag{20}$$

By virtue of (19), (18) and (20), we deduce that $(\varphi - \psi^0_{j_0})$ reaches a local maximum on the curve

$$\begin{aligned}[]
[\varepsilon_0, 1] \ni t \to \big[&\eta_1 t^{\alpha_0}, \dots, \eta_1 t^{\alpha_{n_1-1}}; \dots; \eta_{i-1} t^{\alpha_{n_1+\cdots+n_{i-2}}}, \dots, \eta_{i-1} t^{\alpha_{n_1+\cdots+n_{i-1}-1}};\\
&1, t, t^{\alpha_{n_1+\cdots+n_{i-1}+2}}, \dots, t^{\alpha_{n_1+\cdots+n_i-1}} \eta_{i+1} t^{\alpha_{n_1+\cdots+n_i}}, \dots, \eta_{i+1} t^{\alpha_{n_1+\cdots+n_{i+1}-1}};\\
&\dots; \eta_p t^{\alpha_{n_1+\cdots+n_{p-1}}}, \dots, \eta_p t^{\alpha_{n_1+\cdots+n_p-1}}\big]
\end{aligned}$$

which crosses P_{ε_0} at $t = \varepsilon_0$, then P_0 at $t = x^0_{n_1+\cdots+n_{i-1}+1}$ and, finally, P_1 at $t = 1$. This is in contradiction with the admissibility of φ. □

3.2. *Proof of Theorem 2*

Let $\varphi \in C^\infty(\mathbb{P}_m\mathbb{C})$ be a g-admissible and G-invariant function satisfying $\sup \varphi = 0$ on $\mathbb{P}_m\mathbb{C}$. According to Theorem 1, we have $\varphi \geqslant \psi$. This implies that, for a positive α,

$$\int_{\mathbb{P}_m\mathbb{C}} \exp(-\alpha\varphi)\, dv \leqslant \int_{\mathbb{P}_m\mathbb{C}} \exp(-\alpha\psi)\, dv.$$

We now determine the values of α for which the second integral exists. To this end, we estimate $\int_{\mathbb{P}_m\mathbb{C}} \exp(-\alpha\psi_k)\, dv$ in the chart $\{z_0 = 1\}$. In this chart, the volume element is given by

$$dv = (i)^m \frac{dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_m \wedge d\bar z_m}{(1 + |z_1|^2 + \cdots + |z_m|^2)^{m+1}}.$$

Since the ψ_k depend only on $|z_j|$, the change of variables $u_j = |z_j|^2$ gives

$$\int_{\mathbb{P}_m\mathbb{C}} \exp(-\alpha\psi_k)\, dv$$

$$= \text{Const.} \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{(u_{n_1+\cdots+n_{k-1}} \cdots u_{n_1+\cdots+n_k-1})^{-\alpha(m+1)/n_k}\, du_1 \cdots du_m}{(1 + u_1 + \cdots + u_m)^{(1-\alpha)(m+1)}}.$$

The convergence at zero requires the condition $1 - \frac{\alpha(m+1)}{n_k} > 0$, i.e. $\alpha < n_k/(m+1)$.

At infinity, we use spherical coordinates, and are let to study the convergence of

$$\int_{a>0}^{\infty} r^{-\alpha(m+1)} r^{(\alpha-1)(m+1)} r^{m-1}\, dr = \int_{a>0}^{\infty} r^{-2}\, dr,$$

which converges for all values of α. In conclusion, $\int_{\mathbb{P}_m\mathbb{C}} \exp(-\alpha\psi)\, dv$ exists for $\alpha < \inf_{1 \leqslant k \leqslant p}(\frac{n_k}{m+1})$.

3.3. Proof of Theorem 3

We shall use the invariance of the functions $\varphi([Z_1,\dots,Z_p],[\zeta_0,\dots,\zeta_{n_1-1}],\dots,[\zeta_{n_1+\dots+n_{p-1}},\dots,\zeta_{n_1+\dots+n_p-1}=\zeta_m])$ where $Z_k=(z_{n_1+\dots+n_{k-1}},\dots,z_{n_1+\dots+n_k-1})$ and $(\zeta_{n_1+\dots+n_{k-1}},\dots,\zeta_{n_1+\dots+n_k-1})$ are proportional. The steps of the proof are exactly the same as those of Theorem 1; by means of analogous preliminary lemmas, proved in the same way, we obtain that

$$\begin{aligned}
&(\varphi-\tilde{\psi})\big([1,x_1,\dots,x_m],[1,x_1,\dots,x_{n_1-1}],\dots,[x_{n_1+\dots+n_{p-1}},\dots,x_m]\big)\\
&\quad\geqslant(\varphi-\tilde{\psi})\big(\big[1,x_1,\dots,x_{n_1-1};\zeta_2^{[n_2]};\dots;\zeta_p^{[n_p]}\big],[1,x_1,\dots,x_{n_1-1}],\big[1^{[n_2]}\big];\\
&\qquad\dots;\big[1^{[n_p]}\big]\big),
\end{aligned}\tag{21}$$

where $\zeta_k=(x_{n_1+\dots+n_{k-1}}\cdots x_{n_1+\dots+n_k-1})^{1/n_k}$ for $2\leqslant k\leqslant p$ and $\zeta_k^{[l]}=(\zeta_k,\dots,\zeta_k)\in\mathbb{C}^l$. Then,

$$\begin{aligned}
&(\varphi-\tilde{\psi})\big([x_0,x_1,\dots,x_{n_1-1};1,x_{n_1+1},\dots,x_m],[x_0,x_1,\dots,x_{n_1-1}],\\
&\qquad[1,x_{n_1+1},\dots,x_{n_1+n_2-1}],\dots,[x_{n_1+\dots+n_{p-1}},\dots,x_m]\big)\\
&\quad\geqslant(\varphi-\tilde{\psi})\big([\eta,\eta,\dots,\eta;1,x_{n_1+1},\dots,x_m],\big[1^{[n_1]}\big],\\
&\qquad[1,x_{n_1+1},\dots,x_{n_1+n_2-1}],\dots,[x_{n_1+\dots+n_{p-1}},\dots,x_m]\big),
\end{aligned}\tag{22}$$

where $\eta=(x_0x_1\cdots x_{n_1-1})^{1/n_1}$. As in Lemma 3, we prove that

$$\begin{aligned}
&(\varphi-\tilde{\psi})\big([1,x_1,\dots,x_{n_1-1};x_{n_1},x_{n_1+1},\dots,x_m],[1,x_1,\dots,x_{n_1-1}],\dots,\\
&\qquad[x_{n_1+\dots+n_{p-1}},\dots,x_m]\big)\\
&\quad\geqslant(\varphi-\tilde{\psi})\big(\big[1^{[n_1]};\eta_2^{[n_2]},\dots,\eta_p^{[n_p]}\big],\big[1^{[n_1]}\big],\dots,\big[1^{[n_p]}\big]\big),
\end{aligned}\tag{23}$$

where $\eta_i=\frac{\zeta_i}{\eta}$. Finally, using appropriate auxiliary functions on M, we prove that

$$(\varphi-\tilde{\psi})\big(\big[1^{[n_1]};\eta_2^{[n_2]},\dots,\eta_p^{[n_p]}\big],\big[1^{[n_1]}\big],\dots,\big[1^{[n_p]}\big]\big)\geqslant 0.\tag{24}$$

As a corollary, we obtain that

$$\int_M \exp(-\alpha\varphi)\,d\tilde{v}\leqslant\int_M\exp(-\alpha\tilde{\psi})\,d\tilde{v},$$

and this last integral converges for $\alpha<1/p$.

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